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N-dimensional superintegrable systems from symplectic realizations of Lie coalgebras

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Abstract

The construction of N-dimensional (ND) integrable systems from coalgebras is reviewed. In the case of Poisson coalgebras, a necessary condition for the integrability of the ND coalgebra Hamiltonian coming from a given coalgebra is obtained in terms of the dimension of the symplectic realization and the number of nonlinear Casimir functions. From this viewpoint, the full set of three-, four- and five-dimensional Lie–Poisson coalgebras is analysed, and many new families of multiparametric ND integrable systems coming from the cases that fulfil the integrability condition are obtained, including the explicit form of the integrals of the motion. The superintegrability of these Hamiltonians is also emphasized, and the generalization of the whole construction to the quantum mechanical case is straightforward.

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1. Introduction

Coalgebras are either Poisson or commutator algebras endowed with a compatible comultiplication structure, and have been recently shown to be the 'hidden' symmetries underlying the integrability properties of a wide class of N-dimensional (ND) (super)integrable classical and quantum Hamiltonian systems (see [1–10] and references therein). In this construction, once a symplectic (respectively operatorial) realization of the coalgebra is given, their generators play the role of dynamical symmetries of the Hamiltonian, whilst the coproduct map of the coalgebra is used to 'propagate' the integrability to arbitrary dimension.

From this coalgebra approach, several well-known (super)integrable systems have been recovered and some integrable deformations for them, as well as new integrable systems, have also been obtained. In particular, this integrability-preserving coalgebra deformation procedure has been used to introduce both superintegrable and integrable-free motions on spaces with either constant or variable curvature, and (super)integrable potential terms can also be considered on such spaces [7–10].

In this paper, we get a deeper and more systematic viewpoint to this algebraic approach to Hamiltonian integrability, where the role of symplectic realizations of Poisson coalgebras is emphasized. In particular, we perform a detailed analysis of three-, four- and five-dimensional Lie coalgebras that admit symplectic realizations such that the latter lead to new infinite families of completely integrable (in fact, superintegrable) Hamiltonian systems with N degrees of freedom. Such systems are fully constructed, together with the explicit form of their integrals of the motion. In this way, we present a more global perspective on both the possibilities and limitations of the coalgebra symmetry approach.

The structure of the paper is as follows. In section 2, the construction of ND Hamiltonian systems endowed with coalgebra symmetry is reviewed. In section 3, a necessary condition for the complete integrability of the Hamiltonian in terms of the symplectic realization and the number of nonlinear Casimirs of the coalgebra is found. The intrinsic superintegrability properties of the coalgebra approach are also summarized. The following sections are devoted to the systematic construction of ND integrable Hamiltonians for all the three-, four- and five-dimensional Lie–Poisson algebras whose symplectic realizations fulfil the integrability condition. We emphasize that many of these integrable systems are here introduced for the first time. A closing section including some remarks and comments ends the paper.

2. Hamiltonian systems on Poisson coalgebras

We recall that a coalgebra (A, Δ) is a (unital, associative) algebra A endowed with a coproduct map [11]:

$$\Delta : A \to A \otimes A, \tag{2.1}$$

which is coassociative

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta. \tag{2.2}$$

In addition, Δ has to be an algebra homomorphism from A to $A \otimes A$:

$$\Delta(ab) = \Delta(a)\Delta(b), \qquad \forall\, a, b \in A. \tag{2.3}$$

Moreover, if A is a Poisson algebra and

$$\Delta(\{a, b\}_A) = \{\Delta(a), \Delta(b)\}_{A\otimes A}, \qquad \forall\, a, b \in A, \tag{2.4}$$

we shall say that (A, Δ) is a Poisson coalgebra.

Let us summarize the general construction of [1]. Let (A, Δ) be a Poisson coalgebra with l generators X_i, $(i = 1, \ldots, l)$, and r is the number of functionally independent Casimir functions $\mathcal{C}_j(X_1, \ldots, X_l)$ (with $j = 1, \ldots, r$). Among them, a certain subset of R Casimir functions will be nonlinear functions of the generators of the coalgebra, and will be the relevant ones as far as integrability is concerned.

The coassociative coproduct $\Delta \equiv \Delta^{(2)}$ has to be a Poisson map with respect to the usual Poisson bracket on $A \otimes A$:

$$\{X_i \otimes X_j, X_r \otimes X_s\}_{A\otimes A} = \{X_i, X_r\}_A \otimes X_j X_s + X_i X_r \otimes \{X_j, X_s\}_A. \tag{2.5}$$

Then, the mth coproduct map $\Delta^{(m)}(X_i)$

$$\Delta^{(m)} : A \to A \otimes A \otimes \cdots^{m)} \otimes A, \tag{2.6}$$

can be defined by applying recursively the coproduct $\Delta^{(2)}$ in the form

$$\Delta^{(m)} := (\mathrm{id} \otimes \mathrm{id} \otimes \cdots^{m-2)} \otimes \mathrm{id} \otimes \Delta^{(2)}) \circ \Delta^{(m-1)}. \tag{2.7}$$

Such an induction ensures that $\Delta^{(m)}$ is also a Poisson map.

Table 1. Functions obtained by applying the coproduct map.

X_1	X_2	$\cdots$	X_l	$\mathcal{C}_1$	$\mathcal{C}_2$	$\cdots$	$\mathcal{C}_r$
$\Delta^{(2)}(X_1)$	$\Delta^{(2)}(X_2)$	$\cdots$	$\Delta^{(2)}(X_l)$	$\Delta^{(2)}(\mathcal{C}_1)$	$\Delta^{(2)}(\mathcal{C}_2)$	$\cdots$	$\Delta^{(2)}(\mathcal{C}_r)$
$\Delta^{(3)}(X_1)$	$\Delta^{(3)}(X_2)$	$\cdots$	$\Delta^{(3)}(X_l)$	$\Delta^{(3)}(\mathcal{C}_1)$	$\Delta^{(3)}(\mathcal{C}_2)$	$\cdots$	$\Delta^{(3)}(\mathcal{C}_r)$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\Delta^{(N)}(X_1)$	$\Delta^{(N)}(X_2)$	$\cdots$	$\Delta^{(N)}(X_l)$	$\Delta^{(N)}(\mathcal{C}_1)$	$\Delta^{(N)}(\mathcal{C}_2)$	$\cdots$	$\Delta^{(N)}(\mathcal{C}_r)$

In this way, we can construct the set of functions shown in table 1. From them, given a smooth function $\mathcal{H}(X_1, \ldots, X_l)$, the N-sites Hamiltonian is defined as the Nth coproduct of $\mathcal{H}$:

$$H^{(N)} := \Delta^{(N)}(\mathcal{H}(X_1, \ldots, X_l)) = \mathcal{H}(\Delta^{(N)}(X_1), \ldots, \Delta^{(N)}(X_l)). \tag{2.8}$$

From [1] it can be proven that the set of $r \cdot N$ functions ($m = 1, \ldots, N$; $j = 1, \ldots, r$)

$$C_j^{(m)} := \Delta^{(m)}(\mathcal{C}_j(X_1, \ldots, X_l)) = \mathcal{C}_j(\Delta^{(m)}(X_1), \ldots, \Delta^{(m)}(X_l)), \tag{2.9}$$

Poisson-commute with the Hamiltonian

$$\left\{C_j^{(m)}, H^{(N)}\right\}_{A\otimes A\otimes\cdots^{N)}\otimes A} = 0, \tag{2.10}$$

and is in involution:

$$\left\{C_i^{(m)}, C_j^{(n)}\right\}_{A\otimes A\otimes\cdots^{N)}\otimes A} = 0, \qquad m, n = 1, \ldots, N, \qquad i, j = 1, \ldots, r. \tag{2.11}$$

This construction is completely general, and holds when the Poisson bracket is replaced by the commutator, thus providing an immediate generalization of this approach to the case of quantum mechanical systems, for which the usual ordering prescriptions are taken into account.

There are two main classes of coalgebras to be used to generate ND integrable systems through this procedure. The first one are the Lie–Poisson algebras g^* with generators $X_i (i = 1, \ldots, l)$ and Casimir functions $\mathcal{C}_j(X_1, \ldots, X_l) (j = 1, \ldots, r)$, which are always endowed with the (primitive) coalgebra structure:

$$\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i. \tag{2.12}$$

Then, a very natural choice is to consider Hamiltonians $\mathcal{H}(X_1, \ldots, X_l)$ that, under the iterated application of the coproduct map, will generate dynamical systems on $g^* \otimes g^* \otimes \cdots^{N)} \otimes g^*$ with constants of the motion in involution coming from the nonlinear Casimir functions. These will be the kind of systems that will be systematically explored in this paper.

The second one are the Poisson analogues of quantum algebras and groups [11], which are also (deformed) coalgebras (A_z, Δ_z) (where z is the deformation parameter). Consequently, any function of the generators of a given 'quantum' Poisson algebra (with deformed Casimir elements $C_{z,j}$) will provide a deformation of the Hamiltonian constructed in terms of the undeformed Lie–Poisson coalgebra. Some of these systems have been already explored in the previous literature (for instance, in [1, 2, 7]), and we stress that all the systems presented here can be deformed without altering their integrability properties provided that appropriate coalgebra deformations are constructed. In this respect, we stress that the problem of the classification of quantum deformations is only fully solved for all 3D Lie algebras and for some isolated cases in slightly higher dimensions (see [3] and references therein).

3. Symplectic realizations and complete integrability

For any l-dimensional Poisson coalgebra (A, Δ) and for any smooth Hamiltonian function $\mathcal{H}$ depending on l variables, we can construct a Hamiltonian system on the Poisson manifold $A^{\otimes N}$ constructed as N-tensor copies of A. This is a 'cluster-type' dynamical system [5] with $l \cdot N$ dynamical variables whose evolution equations are

$$\dot{X}_{(i,m)} = \{X_{(i,m)}, H^{(N)}\}, \qquad i = 1, \ldots, l, \qquad m = 1, \ldots, N, \tag{3.1}$$

where $X_{(i,m)}$ denotes the generator X_i living on the mth copy of A. The r Casimir functions of A generate a maximum number of $r \cdot N$ integrals of the motion for $H^{(N)}$ (see table 1) but since, in general, $l - r \geqslant 2$ we have always less than $l \cdot N - 1$ integrals and, therefore, complete integrability for $\mathcal{H}$ cannot be reached in terms of the 'algebraic' dynamical variables $X_{(i,m)}$.

However, we can get a specialization of the coalgebra formalism by working on the symplectic leaves of the initial Poisson coalgebra A, that can be parametrized through suitable symplectic realizations. If A has r independant Casimir functions, a symplectic leaf of A (always even dimensional) will be denoted by $A_{(k_1,k_2,\ldots,k_r)}$, where the leaf is characterized by a given set of constant values $(k_1, k_2, \ldots, k_r)$ for the Casimir functions.

An s-dimensional *symplectic realization* D for $A_{(k_1,k_2,\ldots,k_r)}$ is given (locally) in terms of s pairs (q_i, p_i) of canonical Darboux variables

$$D : x \to x(q_1, p_1, q_2, p_2, \ldots, q_s, p_s), \tag{3.2}$$

where x is any point on $A_{(k_1,k_2,\ldots,k_r)}$. Different symplectic leaves $A_{(k_{1,i},k_{2,i},\ldots,k_{r,i})}$ can be chosen for the different copies of A within $A^{\otimes N}$. In this way, $H^{(N)}$ is defined on the Nth tensor product of the symplectic leaves

$$A_{(k_{1,1},k_{2,1},\ldots,k_{r,1})} \otimes A_{(k_{1,2},k_{2,2},\ldots,k_{r,2})} \otimes \cdots \otimes A_{(k_{1,N},k_{2,N},\ldots,k_{r,N})}, \tag{3.3}$$

where $k_{i,m}$ is the value of the ith Casimir for the mth symplectic leaf.

If we consider symplectic realizations with the same dimension s for all the sites in the tensor product chain, $H^{(N)}$ given by (2.8) turns out to be a function of $N \cdot s$ pairs of canonical variables, i.e. it defines a Hamiltonian system with $N \cdot s$ degrees of freedom. Therefore, we need a number of $(N \cdot s - 1)$ independent and globally defined constants of the motion in involution to state that such Hamiltonian defines a completely integrable system.

3.1. Integrability conditions

At this point, we have to characterize the number of integrals that the coalgebra formalism provides. First of all, note that under the symplectic realizations that we are going to consider, the coproducts of linear Casimirs give just a sum of numerical constants with no dynamical information. Thus, as far as integrability properties are concerned, we have to consider only nonlinear Casimirs, and for each of them we get $(N - 1)$ integrals coming from the $2, \ldots, N$ th coproducts (the one-site evaluation of such Casimirs gives, by construction, constants). Since we have R nonlinear Casimirs (hereafter we will assume that $R \geqslant 1$, since coalgebras without any nonlinear Casimir will be excluded), we get a maximum possible number of integrals in involution given by

$$(N - 1) \cdot R. \tag{3.4}$$

In order to get complete integrability we should have enough number of integrals, which means that

$$N \cdot s - 1 \leqslant (N - 1) \cdot R,$$

thus we need that the chosen symplectic realization D of A fulfils

$$s \leqslant R - \frac{R-1}{N}. \tag{3.5}$$

Therefore, the necessary condition for complete integrability is as follows:

- The symplectic realization has to be of the type $s = 1$ for coalgebras with $R = 1$.
- The symplectic realization has to be of the type $s < R$ for coalgebras with $R > 1$.

Obviously, this condition is not sufficient since the functional independence of a $(N \cdot s - 1)$-dimensional subset of integrals has to be explicitly checked in each case. Note that this condition *does not depend explicitly* on the dimension l of the coalgebra and that low values of s are certainly preferred from the integrability viewpoint.

3.2. *Generic symplectic realizations*

Let us now consider a particular type of symplectic realizations whose s is always fixed by the dimension l and the number r of independant Casimir functions of the coalgebra. We shall call 'generic' the symplectic realization with dimension $s = s_m$ given by

$$s_m = \frac{l-r}{2}. \tag{3.6}$$

This symplectic realization is 'generic' in the sense that will incorporate as many parameters $(k_1, \ldots, k_r)$ as the number of Casimir functions of the coalgebra. Certainly, symplectic realizations with $s < s_m$ could exist, but they are not 'generic' in the sense that can give rise to degeneracies in the Casimir functions (for an explicit example, see [4]).

The sufficient condition for integrability on the generic symplectic realization leads to

$$s_m = \frac{l-r}{2} \leqslant R - \frac{R-1}{N}, \tag{3.7}$$

which leads to the final expression

$$l \leqslant (2R + r) - \frac{2}{N}(R-1), \tag{3.8}$$

which gives us the integrability condition in terms of the dimension l and the number R. Thus, we get that the complete integrability for the generic symplectic realization is achieved provided that

- The dimension $l \leqslant 2R + r = 2 + r$ in the case of $R = 1$ coalgebras.
- The dimension $l < 2R + r$ in the case of $R > 1$ coalgebras.

In this work, we will systematically consider all the Lie coalgebras with $l \leqslant 5$ for which their generic symplectic realizations fulfil the integrability condition, and we will construct all the completely integrable systems associated with them. Throughout the paper, we shall follow the Lie algebra classifications and notation summarized in [12].

The following two remarks are in order:

- For *the classical simple Lie algebras*, $R = r$ so we have that for $R = 1$ coalgebras the condition is $l \leqslant 3$ and for $R > 1$ coalgebras we should have that $l < 3R$. Therefore, apart from the rank 1 cases, this result excludes all simple Lie algebras to provide complete integrable systems in the generic symplectic realization.
- Let us consider *any Lie coalgebra in which the generic symplectic realization has* $s_m = 1$. In that case, provided that $R \geqslant 1$, the coalgebra fulfils always the integrability condition under that symplectic realization and integrable systems can always be constructed.

3.3. Coalgebraic superintegrability

Note that the coassociativity condition (2.2) on the comultiplication map Δ provides a 'two-fold way' for the definition of the image of the generators on $A \otimes A \otimes A$, that will be essential as far as superintegrability is concerned. In fact, this 'two-fold way' can be generalized to the case of the Nth coproduct and, instead of (2.7), another recursion relation for the mth coproduct map can be defined [6]:

$$\Delta_R^{(m)} := (\Delta^{(2)} \otimes \mathrm{id} \otimes \cdots^{m-2)} \otimes \mathrm{id}) \circ \Delta_R^{(m-1)}. \tag{3.9}$$

Due to the coassociativity property of the coproduct, this new map will provide exactly the same expressions for the Nth coproduct of any generator. However, if we label from 1 to N the sites of the chain of N copies of A, lower dimensional coproducts $\Delta^{(m)}$ (with $m < N$) will be 'different' in the sense that $\Delta^{(m)}$ will contain objects living on the tensor product space $1 \otimes 2 \otimes \cdots \otimes m$, whilst $\Delta_R^{(m)}$ will be defined on the sites $(N-m+1) \otimes (N-m+2) \otimes \cdots \otimes N$. Therefore, the coalgebra symmetry of a given Hamiltonian gives rise to two 'pyramidal' sets of $r \cdot N$ integrals of the motion in involution that Poisson-commute with $H^{(N)}$ [6]. The 'right set' of integrals gives rise to some degree of superintegrability of the coalgebra-symmetric Hamiltonians which is, at most, quasi-maximal (note that $(2N-3)$ integrals apart from the Hamiltonian is the maximum possible number of functionally independent integrals, since both sets have $\Delta^{(N)}(\mathcal{C}) \equiv \Delta_R^{(N)}(\mathcal{C})$ in common). So, all the ND systems that will be presented in this paper are, by construction, not only integrable, but *quasi-maximally* superintegrable. Moreover, for some specific choice of the Hamiltonian function, the system could even be maximally superintegrable. In this case, the remaining independent integral of the motion has to be found by other methods different from the coalgebra construction.

4. Integrable systems from 3D Lie coalgebras

By following the known classifications summarized in [12], we first consider the set of nine non-isomorphic 3D ($l = 3$) real Lie algebras, all of them with $r = 1$ (note that the generators e_i in [12] are now written as J_i). Their generic symplectic realizations have been computed and are given in table 2.

Only one of these algebras (namely, the Heisenberg algebra $A_{3,1}$) has $R = 0$ (only one linear Casimir). This will be the only case in which the construction does not provide any dynamically relevant constant of the motion, since the Casimir coincides with the central generator J_1 and its mth coproducts are just numerical constants.

The rest of the 3D Lie coalgebras have $R = 1$ and, since $s_m = (3-1)/2 = 1$, we do have complete integrability for all the latter cases. Therefore, the 'one-particle' symplectic realizations given in table 2 provide *automatically* infinite families of ND completely integrable (and in this case, quasi-maximally superintegrable) systems. In table 2, the constant k is just the value that the Casimir C takes under the given symplectic realization. Note that, in many cases, if $k = 0$ we would get a lower dimensional algebra, a case that we do not consider. We also point out that two symplectic realizations with the same value for k can always be related through a canonical transformation. This is the case of the $A_{3,8}^{\alpha}$ algebra, for which four different symplectic realizations are explicitly provided in order to illustrate the multiplicity of apparently different systems that share the same underlying coalgebra symmetry.

In order to get classical Hamiltonian systems defined through real symplectic realizations, we shall not consider in this paper the coalgebra $A_{3,7}^{\alpha}$ whose Casimir function is complex, and we will also restrict the values of the constant k, if needed.

In order to shorten the presentation of the results, for each Lie coalgebra we will give (a) the explicit non-vanishing commutation rules (that have to be understood as Poisson

Table 2. Generic symplectic realizations and Casimirs for 3D Lie–Poisson algebras.

	J_1	J_2	J_3	C	
$A_{3,1}$	k	p	$-kq$	J_1	$k \neq 0$
$A_{3,2}$	$k\,\mathrm{e}^{\frac{p}{k}}$	$p\,\mathrm{e}^{\frac{p}{k}}$	$-kq$	$J_1\,\mathrm{e}^{-\frac{J_2}{J_1}}$	$k \neq 0$
$A_{3,3}$	$\frac{p^2}{2k}$	$\frac{p^2}{2}$	$-\frac{pq}{2}$	$\frac{J_2}{J_1}$	$k \neq 0$
$A_{3,4}$	$k\,\mathrm{e}^{p}$	e^{-p}	$-q$	$J_1 J_2$	$k \neq 0$
$A_{3,5}^{\alpha}$	$\mathrm{e}^{\frac{p}{\alpha}}$	$k\,\mathrm{e}^{p}$	$-\alpha q$	$J_2 J_1^{-\alpha}$	$k \neq 0$
$A_{3,6}$	$\sqrt{k}\cos p$	$\sqrt{k}\sin p$	$-q$	$J_1^2 + J_2^2$	$k > 0$
$A_{3,7}^{\alpha}$	$\sqrt{k}\,\mathrm{e}^{\alpha p}\cos p$	$\sqrt{k}\,\mathrm{e}^{\alpha p}\sin p$	$-q$	$(J_1^2 + J_2^2)\left(\frac{J_1+\mathrm{i}J_2}{J_1-\mathrm{i}J_2}\right)^{\mathrm{i}\alpha}$	
$A_{3,8}$	$\frac{\mathrm{e}^q}{2}(k - 2p^2)$	p	e^{-q}	$2J_2^2 + J_1 J_3 + J_3 J_1$	$\forall\, k$
$(***)$	$\frac{q^2}{2}$	$\frac{pq}{2}$	$-\frac{p^2}{2} + \frac{k}{q^2}$		$\forall\, k, q \neq 0$
$(**)$	$-pq^2 + \sqrt{2k}q$	$pq - \sqrt{\frac{k}{2}}$	p		$k \geqslant 0$
$(*)$	$p\sin q + p$	$p\cos q$	$p\sin q - p$		$k = 0$
$A_{3,9}$	p	$\sqrt{k - p^2}\cos q$	$\sqrt{k - p^2}\sin q$	$J_1^2 + J_2^2 + J_3^2$	$k > 0$

brackets) that are fulfilled by the given symplectic realization, (b) the most general integrable Hamiltonian $H^{(N)}$ that can be constructed as an arbitrary function of the Nth coproduct of the three generators and (c) the explicit form of the constants of the motion $C^{(m)} (m = 2, \ldots, N)$ coming from the Casimir function. The 'right' set of constants of the motion [6] that gives rise to the quasi-maximal superintegrability of all these systems can be obtained from a given set $C^{(m)}$ by performing the appropriate permutation of indices, as explained in [6]. Note that for each Lie coalgebra we obtain an *infinite* family of superintegrable ND Hamiltonians (the function $\mathcal{H}$ is arbitrary) that depend on N *arbitrary constants* k_i that label the specific symplectic realization that we have chosen on each copy of the Lie coalgebra.

• $A_{3,2}$ *integrable systems.*

$$[J_1, J_3] = J_1, \qquad [J_2, J_3] = J_1 + J_2, \tag{4.1}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^{N} k_i\,\mathrm{e}^{\frac{p_i}{k_i}}, \sum_{i=1}^{N} p_i\,\mathrm{e}^{\frac{p_i}{k_i}}, -\sum_{i=1}^{N} k_i q_i\right), \tag{4.2}$$

$$C^{(m)} = \left(\sum_{i=1}^{m} k_i\,\mathrm{e}^{\frac{p_i}{k_i}}\right)\mathrm{e}^{-\left(\frac{\sum_{i=1}^{m} p_i\,\mathrm{e}^{\frac{p_i}{k_i}}}{\sum_{i=1}^{m} k_i\,\mathrm{e}^{\frac{p_i}{k_i}}}\right)}, \qquad m = 2, \ldots, N. \tag{4.3}$$

Note that in this case (and in some of the remaining examples) the integrals do not depend on the canonical coordinates q_i.

• $A_{3,3}$ *integrable systems.*

$$[J_1, J_3] = J_1, \qquad [J_2, J_3] = J_2, \tag{4.4}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^{N} \frac{p_i^2}{2k_i}, \sum_{i=1}^{N} \frac{p_i^2}{2}, -\sum_{i=1}^{N} \frac{p_i q_i}{2}\right), \tag{4.5}$$

$$C^{(m)} = \left(\sum_{i=1}^{m} p_i^2\right)\Bigg/\left(\sum_{i=1}^{m} \frac{p_i^2}{k_i}\right), \qquad m = 2, \ldots, N. \tag{4.6}$$

Surprisingly enough, in this case we lose the integrability if we take the same symplectic realization on all the copies of the A algebra (i.e. if $k_1 = k_2 = \cdots = k_N$), since all the integrals are transformed into constants.

• $A_{3,4}$ *integrable systems.*

$$[J_1, J_3] = J_1, \qquad [J_2, J_3] = -J_2, \tag{4.7}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^{N} k_i\, \mathrm{e}^{p_i}, \sum_{i=1}^{N} \mathrm{e}^{-p_i}, -\sum_{i=1}^{N} q_i\right), \tag{4.8}$$

$$C^{(m)} = \sum_{i=1}^{m} k_i + \sum_{\substack{i,j=1\\ i\neq j}}^{m} k_i\, \mathrm{e}^{p_i - p_j}, \qquad m = 2, \ldots, N. \tag{4.9}$$

This algebra is just the (1 + 1) Poincaré algebra. An analogue of the Ruijsenaars–Schneider Hamiltonian [13] was obtained in [1] by using a quantum deformation of this realization.

• $A_{3,5}^{\alpha}$ *integrable systems.*

$$[J_1, J_3] = J_1, \qquad [J_2, J_3] = \alpha J_2, \qquad (0 < |\alpha| < 1), \tag{4.10}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^{N} \mathrm{e}^{\frac{p_i}{\alpha}}, \sum_{i=1}^{N} k_i\, \mathrm{e}^{p_i}, -\sum_{i=1}^{N} \alpha q_i\right), \tag{4.11}$$

$$C^{(m)} = \frac{\sum_{i=1}^{m} k_i\, \mathrm{e}^{p_i}}{\left(\sum_{i=1}^{m} \mathrm{e}^{\frac{p_i}{\alpha}}\right)^{\alpha}}, \qquad m = 2, \ldots, N. \tag{4.12}$$

• $A_{3,6}$ *integrable systems.*

$$[J_1, J_3] = -J_2, \qquad [J_2, J_3] = J_1, \tag{4.13}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^{N} \sqrt{k_i}\cos p_i, \sum_{i=1}^{N} \sqrt{k_i}\sin p_i, -\sum_{i=1}^{N} q_i\right), \tag{4.14}$$

$$C^{(m)} = \sum_{i=1}^{m} k_i + \sum_{\substack{i,j=1\\ i\neq j}}^{m} \sqrt{k_i k_j}\, \cos(p_i - p_j), \qquad m = 2, \ldots, N. \tag{4.15}$$

Note that the algebra $A_{3,6}$ is the two-dimensional Euclidean algebra.

• $A_{3,8}$ *integrable systems.*

$$[J_1, J_2] = J_1, \qquad [J_1, J_3] = -2J_2, \qquad [J_2, J_3] = J_3, \tag{4.16}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^{N} \frac{\mathrm{e}^{q_i}}{2}(k_i - 2p_i^2), \sum_{i=1}^{N} p_i, \sum_{i=1}^{N} \mathrm{e}^{-q_i}\right), \tag{4.17}$$

$$C^{(m)} = \sum_{i=1}^{m} k_i + \sum_{\substack{i,j=1\\ i\neq j}}^{m} \mathrm{e}^{q_j - q_i}\left(k_j - 2p_j^2\right) + 2\sum_{\substack{i,j=1\\ i\neq j}}^{m} p_i p_j, \qquad m = 2, \ldots, N. \tag{4.18}$$

This is the $sl(2, R) \simeq so(2, 1)$ Lie coalgebra, and many different and important integrable systems can be obtained as $H^{(N)}$ by making use of different symplectic realizations. For instance, the Calogero–Gaudin Hamiltonian [14, 15]

$$H^{(N)} = \sum_{i<j}^{N} 2p_i p_j (1 - \cos(q_i - q_j)) \tag{4.19}$$

comes from the $k = 0$ symplectic realization $(*)$ of $A_{3,8}$ (the $sl(2)$ algebra) by taking as the Hamiltonian the Casimir operator C (see [1, 16] for a detailed discussion of this system and its integrable deformations, and papers [17, 18] for its q-deformed quantum mechanical version). In general, note that the choice of the symplectic realization drastically changes the 'shape' of the Hamiltonian. For example, by using the Gel'fand–Dyson symplectic map $(**)$ with $k = 0$, the very same Calogero–Gaudin system reads

$$H^{(N)} = \sum_{i<j}^{N} -p_i p_j (q_i - q_j)^2. \tag{4.20}$$

Moreover, the following Hamiltonian

$$H^{(N)} = \sum_{i=1}^{N} \left(\frac{p_i^2}{2} - \frac{k_i}{q_i^2} \right) + \mathcal{F}\left(\sum_{i=1}^{N} q_i^2 \right) \tag{4.21}$$

represents the motion of a particle on the ND Euclidean space under the action of N 'centrifugal barriers' determined by the k_i terms and an arbitrary central potential given by the function $\mathcal{F}$. This is also $A_{3,8}$ coalgebra-invariant under the symplectic realization $(***)$ [2] of table 2 provided the Hamiltonian function is taken as

$$\mathcal{H} = -J_3 + \mathcal{F}(2J_1).$$

Therefore, as particular cases, this Hamiltonian reproduces the Smorodinsky–Winternitz system [19] for $\mathcal{F} = 2\omega J_1$ and provides a ND generalization of the Kepler potential when $\mathcal{F} = -\gamma/\sqrt{2J_1}$ (ω and γ are real constants). In fact, this $A_{3,8}$ coalgebra has been recently shown to underlie the integrability of the oscillator and Kepler potentials on the ND spaces with constant curvature [9] and also of the ND spherically symmetric generalization of certain spaces with non-constant curvature [10].

- $A_{3,9}$ *integrable systems.*

$$[J_1, J_2] = J_3, \qquad [J_1, J_3] = -J_2, \qquad [J_2, J_3] = J_1, \tag{4.22}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^{N} p_i, \sum_{i=1}^{N} \sqrt{k_i - p_i^2} \cos q_i, \sum_{i=1}^{N} \sqrt{k_i - p_i^2} \sin q_i \right), \tag{4.23}$$

$$C^{(m)} = -\sum_{i=1}^{m} k_i + 2\sum_{i=1}^{m} p_i^2 + \sum_{\substack{i,j,\\ i\neq j}}^{m} p_i p_j + \sum_{\substack{i,j,\\ i\neq j}}^{m} \cos(q_i - q_j)\sqrt{p_i^2 - k_i}\sqrt{p_j^2 - k_j},$$
$$m = 2, \ldots, N. \tag{4.24}$$

These would be the classical integrable systems provided by N copies of the $so(3)$ algebra.

5. Integrable systems from 4D Lie coalgebras

In this case, the classification [12] provide a set of 12 non-isomorphic 4D ($l = 4$) real Lie algebras. Among them, four algebras have $R = 0$, and will not give rise to integrable systems.

Table 3. Symplectic realizations for 4D Lie–Poisson algebras with $R = 1$.

	J_1	J_2	J_3	J_4	C_i	
$A_{4,1}$	k_1	p	$\frac{p^2-k_2}{2k_1}$	$-k_1 q$	$C_2 = J_2^2 - 2J_1J_3$	$k_1 \neq 0$
$A_{4,3}$	e^p	k_1	$k_1\,(p - \log k_2)$	$-q$	$C_2 = J_1\,e^{\frac{-J_3}{J_2}}$	$k_1 \neq 0, k_2 > 0$
$A_{4,8}$	k_1	$\sqrt{p}\,e^q$	$k_1\sqrt{p}\,e^{-q}$	$-\frac{k_2}{2k_1} + p$	$C_2 = J_2J_3 + J_3J_2 - 2J_1J_4$	$k_1 \neq 0$
$A_{4,10}$	k_1	$k_1 q$	p	$\frac{-p^2-k_1^2q^2+k_2}{2k_1}$	$C_2 = 2J_1J_4 + J_2^2 + J_3^2$	$k_1 \neq 0$

5.1. Algebras with $R = 1$

We have four Lie algebras with $r = 2$ and $R = 1$ (two Casimirs, one of them linear). In all these cases $s_m = (4-2)/2 = 1$, and the integrability condition is fulfilled. The explicit generic symplectic realizations of these four algebras, together with the explicit form of the nonlinear Casimir, are given in table 3.

• $A_{4,1}$ *integrable systems.*

$$[J_2, J_4] = J_1, \qquad [J_3, J_4] = J_2, \tag{5.1}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^N k_{1,i}, \sum_{i=1}^N p_i, \sum_{i=1}^N \frac{p_i^2 - k_{2,i}}{2k_{1,i}}, -\sum_{i=1}^N k_{1,i}q_i\right), \tag{5.2}$$

$$C^{(m)} = \sum_{i=1}^m k_{2,i} + \sum_{\substack{i,j=1\\ i\neq j}}^m p_ip_j + \sum_{\substack{i,j=1\\ i\neq j}}^m k_{1,i}\frac{k_{2,j}}{k_{1,j}} - \sum_{\substack{i,j=1\\ i\neq j}}^m \frac{p_i^2}{k_{1,i}}k_{1,j}, \qquad m = 2,\ldots,N. \tag{5.3}$$

The algebra $A_{4,1}$ is the (1+1) extended Galilei Lie algebra, and their associated integrable systems have been constructed in [3], as well as their integrable deformations.

• $A_{4,3}$ *integrable systems.*

$$[J_1, J_4] = J_1, \qquad [J_3, J_4] = J_2, \tag{5.4}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^N e^{p_i}, \sum_{i=1}^N k_{1,i}, \sum_{i=1}^N k_{1,i}(p_i - \log k_{2,i}), -\sum_{i=1}^N q_i\right), \tag{5.5}$$

$$C^{(m)} = \left(\sum_{i=1}^m e^{p_i}\right) e^{-\frac{\sum_{i=1}^m k_{1,i}(p_i - \log k_{2,i})}{\sum_{i=1}^m k_{1,i}}}, \qquad m = 2,\ldots,N. \tag{5.6}$$

• $A_{4,8}$ *integrable systems.*

$$[J_2, J_3] = J_1, \qquad [J_2, J_4] = J_2, \qquad [J_3, J_4] = -J_3, \tag{5.7}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^N k_{1,i}, \sum_{i=1}^N \sqrt{p_i}\,e^{q_i}, \sum_{i=1}^N k_{1,i}\sqrt{p_i}\,e^{-q_i}, \sum_{i=1}^N\left(-\frac{k_{2,i}}{2k_{1,i}} + p_i\right)\right), \tag{5.8}$$

$$C^{(m)} = \sum_{i=1}^m k_{2,i} + \sum_{\substack{i,j=1\\ i\neq j}}^m \frac{k_{2,i}}{k_{1,i}}k_{1,j} - 2\sum_{\substack{i,j=1\\ i\neq j}}^m p_ik_{1,j} + \sum_{\substack{i,j=1\\ i\neq j}}^m e^{-q_i+q_j}k_{1,i}\sqrt{p_ip_j},$$
$$m = 2,\ldots,N. \tag{5.9}$$

The algebra $A_{4,8}$ is also known as the oscillator algebra h_4. If we take the $\mathcal{H}$ function

$$\mathcal{H} = \lambda J_4 + \mu J_2 J_3, \tag{5.10}$$

under the realization with $k_{1,i} = 1$ and $k_{2,i} = 0$, we get the following integrable Hamiltonian

$$H^{(N)} = (\lambda + \mu) \sum_{i=1}^{N} p_i + 2\mu \sum_{i<j}^{N} \sqrt{p_i p_j} \cosh(q_i - q_j), \tag{5.11}$$

whose quantum mechanical version was introduced in [20]. The integrals of the motion in involution in the chosen realization read

$$C^{(m)} = -2 \sum_{i=1}^{m} p_i + 2 \sum_{i,j=1}^{m} \sqrt{p_i p_j} \cosh(q_i - q_j) = 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \sqrt{p_i p_j} \cosh(q_i - q_j),$$
$$m = 2, \ldots, N. \tag{5.12}$$

• $A_{4,10}$ *integrable systems.*

$$[J_2, J_3] = J_1, \qquad [J_2, J_4] = -J_3, \qquad [J_3, J_4] = J_2, \tag{5.13}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^{N} k_{1,i}, \sum_{i=1}^{N} k_{1,i} q_i, \sum_{i=1}^{N} p_i, -\sum_{i=1}^{N} \frac{p_i^2 + k_{1,i}^2 q_i^2 - k_{2,i}}{2k_{1,i}} \right), \tag{5.14}$$

$$C^{(m)} = \sum_{i,j=1}^{m} \frac{k_{2,i} k_{1,j}}{k_{1,i}} - \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \frac{p_i^2 k_{1,j}}{k_{1,i}} + \sum_{\substack{i,j=1 \\ i \neq j}}^{m} k_{1,i} k_{1,j} q_i q_j + \sum_{\substack{i,j=1 \\ i \neq j}}^{m} p_i p_j - \sum_{\substack{i,j=1 \\ i \neq j}}^{m} q_i^2 k_{1,i} k_{1,j},$$
$$m = 2, \ldots, N. \tag{5.15}$$

In particular, the N-particle Hamiltonian given by $\mathcal{H} = -J_4 + F(J_2)$ gives

$$H^{(N)} = \sum_{i=1}^{N} \frac{p_i^2 + k_{1,i}^2 q_i^2 - k_{2,i}}{2k_{1,i}} + F\left(\sum_{i=1}^{N} k_{1,i} q_i \right) \tag{5.16}$$

which is a new completely integrable Hamiltonian for any choice of the function F, with integrals of the motion independent of F and given by (5.15).

5.2. Algebras with $R = 2$

We have four more 4D algebras with $R = 2$. Again, $s_m = (4-2)/2 = 1$ and the integrability condition is fulfilled. One of them (namely, $A_{4,6}^{a,b}$) has a complex invariant, so we discard it. The generic symplectic realization of the three remaining ones are given in table 4. Note that in all these cases the Casimir functions depend only on the momenta p. More importantly, since we have two nonlinear Casimirs, we obtain two sets $\{C_1^{(m)}\}$ and $\{C_2^{(m)}\}$ of $(N-1)$ constants of the motion in involution with the Hamiltonian, besides the two additional sets given by the 'right' coproducts.

• $A_{4,2}^{\alpha}$ *integrable systems.*

$$[J_1, J_4] = \alpha J_1, \qquad [J_2, J_4] = J_2, \qquad [J_3, J_4] = J_2 + J_3 \qquad (\alpha \neq 0), \tag{5.17}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^{N} \frac{k_{1,i}^{\alpha}}{k_{2,i}} \mathrm{e}^{p_i}, \sum_{i=1}^{N} k_{1,i} \mathrm{e}^{p_i}, \sum_{i=1}^{N} k_{1,i} p_i \mathrm{e}^{p_i}, -\sum_{i=1}^{N} q_i \right), \tag{5.18}$$

Table 4. Symplectic realizations for 4D Lie–Poisson algebras with $R = 2$.

	J_1	J_2	J_3	J_4	C_i	
$A_{4,2}^{\alpha}$	$\frac{k_1^{\alpha}}{k_2}\,\mathrm{e}^{p}$	$k_1\,\mathrm{e}^{p}$	$k_1 p\,\mathrm{e}^{p}$	$-q$	$C_1 = J_2\,\mathrm{e}^{-\frac{J_3}{J_2}}, C_2 = \frac{J_2^{\alpha}}{J_1}$	$k_1 \neq 0$
$A_{4,4}$	$k_1\,\mathrm{e}^{p}$	$k_1 p\,\mathrm{e}^{p}$	$\frac{k_1\,\mathrm{e}^{p}}{2}(p^2+k_2)$	$-q$	$C_1 = J_1\,\mathrm{e}^{-\frac{J_2}{J_1}}, C_2 = \frac{2J_1J_3-J_2^2}{J_1^2}$	$k_1 \neq 0$
$A_{4,5}^{a,b}$	$(k_1a)^{\frac{1}{a}}\,\mathrm{e}^{p}$	$a\,\mathrm{e}^{ap}$	$\frac{(k_1a)^{\frac{b}{a}}}{k_2}\,\mathrm{e}^{bp}$	$-q$	$C_1 = \frac{J_1^a}{J_2}, C_2 = \frac{J_1^b}{J_3}$	$k_1 \neq 0$

$$C_1^{(m)} = \left(\sum_{i=1}^{m} k_{1,i}\,\mathrm{e}^{p_i}\right)\mathrm{e}^{-\left(\frac{\sum_{i=1}^m k_{1,i}p_i\,\mathrm{e}^{p_i}}{\sum_{i=1}^m k_{1,i}\,\mathrm{e}^{p_i}}\right)}, \qquad C_2^{(m)} = \frac{\left(\sum_{i=1}^m k_{1,i}\,\mathrm{e}^{p_i}\right)^{\alpha}}{\left(\sum_{i=1}^m \frac{k_{1,i}^{\alpha}}{k_{2,i}}\,\mathrm{e}^{p_i}\right)}, \quad m = 2,\ldots,N. \tag{5.19}$$

• $A_{4,4}$ *integrable systems.*

$$[J_1, J_4] = J_1, \qquad [J_2, J_4] = J_1 + J_2, \qquad [J_3, J_4] = J_2 + J_3, \tag{5.20}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^N k_{1,i}\,\mathrm{e}^{p_i}, \sum_{i=1}^N k_{1,i}p_i\,\mathrm{e}^{p_i}, \sum_{i=1}^N \frac{k_{1,i}\,\mathrm{e}^{p_i}}{2}(p_i^2 + k_{2,i}), -\sum_{i=1}^N q_i\right), \tag{5.21}$$

$$C_1^{(m)} = \left(\sum_{i=1}^m k_{1,i}\,\mathrm{e}^{p_i}\right)\mathrm{e}^{-\frac{\sum_{i=1}^m k_{1,i}p_i\,\mathrm{e}^{p_i}}{\sum_{i=1}^m k_{1,i}\,\mathrm{e}^{p_i}}}, \qquad m = 2,\ldots,N, \tag{5.22}$$

$$C_2^{(m)} = \frac{2\left(\sum_{i=1}^m k_{1,i}\,\mathrm{e}^{p_i}\right)\left(\sum_{i=1}^m \frac{k_{1,i}\,\mathrm{e}^{p_i}}{2}\left(p_i^2 + k_{2,i}\right)\right) - \left(\sum_{i=1}^m k_{1,i}p_i\,\mathrm{e}^{p_i}\right)^2}{\left(\sum_{i=1}^m k_{1,i}\,\mathrm{e}^{p_i}\right)^2}, \quad m = 2,\ldots,N. \tag{5.23}$$

• $A_{4,5}^{a,b}$ *integrable systems.*

$$[J_1, J_4] = J_1, \qquad [J_2, J_4] = aJ_2, \qquad [J_3, J_4] = bJ_3, \qquad ab \neq 0, \quad -1 \leqslant a \leqslant b \leqslant 1, \tag{5.24}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^N (k_{1,i}a)^{\frac{1}{a}}\,\mathrm{e}^{p_i}, \sum_{i=1}^N a\,\mathrm{e}^{ap_i}, \sum_{i=1}^N \frac{(k_{1,i}a)^{\frac{b}{a}}}{k_{2,i}}\,\mathrm{e}^{bp_i}, -\sum_{i=1}^N q_i\right), \tag{5.25}$$

$$C_1^{(m)} = \frac{\left(\sum_{i=1}^m (k_{1,i}a)^{\frac{1}{a}}\,\mathrm{e}^{p_i}\right)^a}{\left(a\sum_{i=1}^m \mathrm{e}^{ap_i}\right)}, \qquad C_2^{(m)} = \frac{\left(\sum_{i=1}^m (k_{1,i}a)^{\frac{1}{a}}\,\mathrm{e}^{p_i}\right)^b}{\left(\sum_{i=1}^m \frac{(k_{1,i}a)^{\frac{b}{a}}}{k_{2,i}}\,\mathrm{e}^{bp_i}\right)}, \qquad m = 2,\ldots,N. \tag{5.26}$$

6. Integrable systems from 5D Lie coalgebras

Non-isomorphic real Lie algebras of dimension five are also fully classified (we use the notation given in [12] for Mubarakzyanov results). In fact, there are 40 different 5D Lie algebras with $r = 1, 3$.

Table 5. Symplectic realizations for 5D Lie–Poisson algebras with $R = 1$.

	J_1	J_2	J_3	J_4	J_5	Nonlinear Casimir	
$A_{5,1}$	k_1	k_2	$k_1 p+\frac{k_3}{k_2}$	$k_2 p$	$-q$	$C_3 = J_2 J_3 - J_1 J_4$	$k_1, k_2 \neq 0$
$A_{5,3}$	k_1	k_2	p	$-k_2 q - \frac{k_3}{2k_1}$	$-k_1 q - \frac{p^2}{2k_2}$	$C_3 = J_3^2 + 2J_2 J_5 - 2J_1 J_4$	$k_1, k_2 \neq 0$

It is easy to check that all the $r = 1$ cases do not fulfil the integrability condition. This is obvious for the seven cases in which $R = 0$. There are also 18 cases with $R = r = 1$, but for them $s_m = (5-1)/2 = 2 > R$.

So, we are left with 15 cases with $r = 3$. For all of them the integrability condition holds, since $R = 1, 2, 3$ and $s_m = (5-3)/2 = 1$. As usual, we will not consider here the five cases that present complex Casimir functions, so we are left with ten new families of integrable systems.

6.1. Algebras with $R = 1$

- *$A_{5,1}$ integrable systems.*

$$[J_3, J_5] = J_1, \qquad [J_4, J_5] = J_2, \tag{6.1}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^{N} k_{1,i}, \sum_{i=1}^{N} k_{2,i}, \sum_{i=1}^{N}\left(k_{1,i} p_i + \frac{k_{3,i}}{k_{2,i}}\right), \sum_{i=1}^{N} k_{2,i} p_i, -\sum_{i=1}^{N} q_i\right), \tag{6.2}$$

$$C_3^{(m)} = \sum_{i=i}^{m} k_{3,i} + \sum_{\substack{i,j=1\\ i\neq j}}^{m} \frac{k_{3,i}k_{2,j}}{k_{2,i}} - \sum_{\substack{i,j=1\\ i\neq j}}^{m} k_{2,i}k_{1,j} p_i + \sum_{\substack{i,j=1\\ i\neq j}}^{m} k_{1,i}k_{2,j} p_i, \qquad m = 2, \ldots, N. \tag{6.3}$$

- *$A_{5,3}$ integrable systems.*

$$[J_3, J_4] = J_2, \qquad [J_3, J_5] = J_1, \qquad [J_4, J_5] = J_3, \tag{6.4}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^{N} k_{1,i}, \sum_{i=1}^{N} k_{2,i}, \sum_{i=1}^{N} p_i, -\sum_{i=1}^{N}\left(k_{2,i} q_i + \frac{k_3}{2k_{1,i}}\right), -\sum_{i=1}^{N}\left(k_{1,i} q_i + \frac{p^2}{2k_{2,i}}\right)\right), \tag{6.5}$$

$$C_3^{(m)} = \sum_{i=i}^{m} k_{3,i} + \sum_{\substack{i,j=1\\ i\neq j}}^{m} p_i p_j + \sum_{\substack{i,j=1\\ i\neq j}}^{m} \frac{k_{3,i}k_{1,j}}{k_{1,i}} - \sum_{\substack{i,j=1\\ i\neq j}}^{m} \frac{k_{2,j}}{k_{2,i}} p_i^2 + 2\sum_{i,j=1}^{m} k_{1,i}k_{2,j}(q_j - q_i), \quad m = 2, \ldots, N. \tag{6.6}$$

The N-particle Hamiltonian $\mathcal{H} = -J_5 + G(-J_4)$ leads to

$$H^{(N)} = \sum_{i=1}^{N} \frac{p_i^2}{2k_{2,i}} + \sum_{i=1}^{N} k_{1,i} q_i + G\left(\sum_{i=1}^{N} k_{2,i} q_i + \sum_{i=1}^{N} \frac{k_{3,i}}{k_{1,i}}\right), \tag{6.7}$$

which is completely integrable for any choice of the function G. In particular, a large family of ND integrable non-homogeneous polynomial potentials are included in this family (see [21] for an exhaustive study of integrable homogeneous polynomial potentials).

Table 6. Symplectic realizations for 5D Lie–Poisson algebras with $R = 2$.

	J_1	J_2	J_3	J_4	J_5	Nonlinear Casimirs	
$A_{5,2}$	k_1	p	$\frac{1}{2k_1}(p^2-k_2)$	$\frac{1}{6k_1^2}(p^3-3k_2p+2k_3)$	$-k_1q$	$C_2=J_2^2-2J_1J_3,\ C_3=J_2^3+3J_1^2J_4-3J_1J_2J_3$	$k_1\neq 0$
$A_{5,8}^c$	k_1	k_1p	$k_3\,\mathrm{e}^p$	$\frac{k_3^c}{k_2}\,\mathrm{e}^{cp}$	$-q$	$C_2=\frac{J_3^c}{J_4},\ C_3=J_3\,\mathrm{e}^{-\frac{J_2}{J_1}}$	$k_1,k_3\neq 0$
$A_{5,10}$	k_1	k_1p	$\frac{k_1}{2}p^2-\frac{k_2}{2k_1}$	$k_3\,\mathrm{e}^p$	$-q$	$C_2=J_2^2-2J_1J_3,\ C_3=J_4\,\mathrm{e}^{-\frac{J_2}{J_1}}$	$k_1,k_3\neq 0$

6.2. Algebras with $R = 2$

• *$A_{5,2}$ Integrable systems.*

$$[J_2, J_5] = J_1, \qquad [J_3, J_5] = J_2, \qquad [J_4, J_5] = J_3, \tag{6.8}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^N k_{1,i}, \sum_{i=1}^N p_i, \sum_{i=1}^N \frac{1}{2k_{1,i}}\left(p_i^2 - k_{2,i}\right), \right. \\ \left. \times \sum_{i=1}^N \frac{1}{6k_{1,i}^2}\left(p_i^3 - 3k_{2,i}p_i + 2k_{3,i}\right), -\sum_{i=1}^N k_{1,i}q_i\right), \tag{6.9}$$

$$C_2^{(m)} = \sum_{i=1}^m k_{2,i} + \sum_{\substack{i,j=1\\ i\neq j}}^m p_i p_j - \sum_{\substack{i,j=1\\ i\neq j}}^m \frac{k_{1,j}}{k_{1,i}}\left(p_i^2 - k_{2,i}\right), \qquad m = 2,\ldots,N, \tag{6.10}$$

$$C_3^{(m)} = \left(\sum_{i=1}^m p_i\right)^3 + 3\left(\sum_{i=1}^m k_{1,i}\right)^2\left(\sum_{i=1}^m \frac{1}{6k_{1,i}^2}\left(p_i^3 - 3k_{2,i}p_i + 2k_{3,i}\right)\right) \\ -3\left(\sum_{i=1}^m k_{1,i}\right)\left(\sum_{i=1}^m p_i\right)\left(\sum_{i=1}^m \frac{1}{2k_{1,i}}\left(p_i^2 - k_{2,i}\right)\right), \qquad m = 2,\ldots,N. \tag{6.11}$$

• *$A_{5,8}^c$ integrable systems.*

$$[J_2, J_5] = J_1, \qquad [J_3, J_5] = J_3, \qquad [J_4, J_5] = cJ_4 \qquad (0 < |c| \leqslant 1), \tag{6.12}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^N k_{1,i}, \sum_{i=1}^N k_{1,i}p_i, \sum_{i=1}^N k_{3,i}\,\mathrm{e}^{p_i}, \sum_{i=1}^N \frac{k_{3,i}^c}{k_{2,i}}\,\mathrm{e}^{cp_i}, -\sum_{i=1}^N q_i\right), \tag{6.13}$$

$$C_2^{(m)} = \frac{\left(\sum_{i=1}^m k_{3,i}\,\mathrm{e}^{p_i}\right)^c}{\left(\sum_{i=1}^m \frac{k_{3,i}^c}{k_{2,i}}\,\mathrm{e}^{cp_i}\right)}, \qquad C_3^{(m)} = \left(\sum_{i=1}^m k_{3,i}\,\mathrm{e}^{p_i}\right)\mathrm{e}^{-\frac{(\sum_{i=1}^m k_{1,i}p_i)}{(\sum_{i=1}^m k_{1,i})}}, \qquad m = 2,\ldots,N. \tag{6.14}$$

• *$A_{5,10}$ integrable systems.*

$$[J_2, J_5] = J_1, \qquad [J_3, J_5] = J_2, \qquad [J_4, J_5] = J_4, \tag{6.15}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^N k_{1,i}, \sum_{i=1}^N k_{1,i}p_i, \sum_{i=1}^N \left(\frac{k_{1,i}}{2}p_i^2 - \frac{k_{2,i}}{2k_{1,i}}\right), \sum_{i=1}^N k_{3,i}\,\mathrm{e}^{p_i}, -\sum_{i=1}^N q_i\right), \tag{6.16}$$

Table 7. Symplectic realizations for 5D Lie–Poisson algebras with $R = 3$.

	J_1	J_2	J_3	J_4	J_5	
$A_{5,7}^{a,b,c}$	e^{p}	$\frac{e^{ap}}{k_1}$	$\frac{e^{bp}}{k_2}$	$\frac{e^{cp}}{k_3}$	$-q$	$k_1, k_2, k_3 \neq 0$
$A_{5,9}^{b,c}$	$k_3\, e^{p}$	$k_3\, p\, e^{p}$	$\frac{k_3^b}{k_1}\, e^{bp}$	$\frac{k_3^c}{k_2}\, e^{cp}$	$-q$	$k_3 \neq 0$
$A_{5,11}^{c}$	$k_2\, e^{p}$	$k_2\, p\, e^{p}$	$\frac{k_2}{2}\, e^{bp}(p^2 + k_3)$	$\frac{k_2^c}{k_1}\, e^{cp}$	$-q$	$k_2 \neq 0$
$A_{5,12}^{c}$	$k_1\, e^{p}$	$k_1\, p\, e^{p}$	$\frac{k_1}{2}\, e^{p}\left(p^2 + k_2\right)$	$\frac{k_1\, e^{p}}{6}\left(2k_3 + 3k_2 p + p^3\right)$	$-q$	$k_1 \neq 0$
$A_{5,15}^{\alpha}$	e^{-p}	$(p - \log k_2)\, e^{p}$	$\frac{e^{\alpha p}}{k_1}$	$\frac{e^{\alpha p}}{\alpha k_1}(\alpha p - \log k_1 k_3)$	$-q$	$k_2 \geqslant 0,\ \frac{k_1}{k_3} > 0$

$$C_2^{(m)} = \sum_{i=1}^{m} k_{2,i} + \sum_{\substack{i,j=1\\ i\neq j}}^{m} \left(\frac{k_{2,i}}{k_{1,i}} - k_{1,i}\, p_i^2 \right) k_{1,j} + \sum_{\substack{i,j=1\\ i\neq j}}^{m} k_{1,i} k_{1,j}\, p_i\, p_j, \qquad m = 2, \ldots, N, \tag{6.17}$$

$$C_3^{(m)} = \left(\sum_{i=1}^{m} k_{3,i}\, e^{p_i} \right) e^{-\frac{(\sum_{i=1}^{m} k_{1,i} p_i)}{(\sum_{i=1}^{m} k_{1,i})}}, \qquad m = 2, \ldots, N. \tag{6.18}$$

6.3. Algebras with $R = 3$

• $A_{5,7}^{a,b,c}$ *integrable systems.*

$$[J_1, J_5] = J_1, \qquad [J_2, J_5] = a J_2, \qquad [J_3, J_5] = b J_3, \qquad [J_4, J_5] = c J_4, abc \neq 0,$$
$$-1 \leqslant c \leqslant b \leqslant a \leqslant 1, \tag{6.19}$$

$$H^{(N)} = \mathcal{H} \left(\sum_{i=1}^{N} e^{p_i}, \sum_{i=1}^{N} \frac{e^{ap_i}}{k_{1,i}}, \sum_{i=1}^{N} \frac{e^{bp_i}}{k_{2,i}}, \sum_{i=1}^{N} \frac{e^{cp_i}}{k_{3,i}}, -\sum_{i=1}^{N} q_i \right). \tag{6.20}$$

The nonlinear Casimirs for this algebra are

$$C_1 = \frac{{J_1}^a}{J_2}, \qquad C_2 = \frac{{J_1}^b}{J_3}, \qquad C_3 = \frac{{J_1}^c}{J_4}, \tag{6.21}$$

$$C_1^{(m)} = \frac{\left(\sum_{i=1}^{m} e^{p_i}\right)^a}{\left(\sum_{i=1}^{m} \frac{e^{ap_i}}{k_{1,i}}\right)}, \qquad C_2^{(m)} = \frac{\left(\sum_{i=1}^{m} e^{p_i}\right)^b}{\left(\sum_{i=1}^{m} \frac{e^{bp_i}}{k_{2,i}}\right)}, \qquad C_3^{(m)} = \frac{\left(\sum_{i=1}^{m} e^{p_i}\right)^c}{\left(\sum_{i=1}^{m} \frac{e^{cp_i}}{k_{3,i}}\right)},$$
$$m = 2, \ldots, N. \tag{6.22}$$

• $A_{5,9}^{b,c}$ *integrable systems.*

$$[J_1, J_5] = J_1, \qquad [J_2, J_5] = J_1 + J_2, \qquad [J_3, J_5] = b J_3, \qquad [J_4, J_5] = c J_4$$
$$(0 \neq c \leqslant b), \tag{6.23}$$

$$H^{(N)} = \mathcal{H} \left(\sum_{i=1}^{N} k_{3,i}\, e^{p_i}, \sum_{i=1}^{N} k_{3,i}\, p_i\, e^{p_i}, \sum_{i=1}^{N} \frac{k_{3,i}^b}{k_{1,i}}\, e^{bp_i}, \sum_{i=1}^{N} \frac{k_{3,i}^c}{k_{2,i}}\, e^{cp_i}, -\sum_{i=1}^{N} q_i \right). \tag{6.24}$$

The nonlinear Casimirs are

$$C_1 = \frac{J_1^b}{J_3}, \qquad C_2 = \frac{J_1^c}{J_4}, \qquad C_3 = J_1\, \mathrm{e}^{-\frac{J_2}{J_1}}, \tag{6.25}$$

$$\begin{aligned} C_1^{(m)} &= \frac{\left(\sum_{i=1}^m k_{3,i}\, \mathrm{e}^{p_i}\right)^b}{\left(\sum_{i=1}^m \frac{k_{3,i}^b}{k_{1,i}}\, \mathrm{e}^{bp_i}\right)}, \qquad C_2^{(m)} = \frac{\left(\sum_{i=1}^m k_{3,i}\, \mathrm{e}^{p_i}\right)^c}{\left(\sum_{i=1}^m \frac{k_{3,i}^c}{k_{2,i}}\, \mathrm{e}^{cp_i}\right)}, \\ C_3^{(m)} &= \left(\sum_{i=1}^m k_{3,i}\, \mathrm{e}^{p_i}\right) \mathrm{e}^{-\frac{(\sum_{i=1}^m k_{3,i}\, p_i\, \mathrm{e}^{p_i})}{(\sum_{i=1}^m k_{3,i}\, \mathrm{e}^{p_i})}}, \qquad m = 2, \ldots, N. \end{aligned} \tag{6.26}$$

- $A_{5,11}^c$ *integrable systems.*

$$[J_1, J_5] = J_1, \quad [J_2, J_5] = J_1 + J_2, \quad [J_3, J_5] = J_2 + J_3, \quad [J_4, J_5] = cJ_4 \quad (c \neq 0), \tag{6.27}$$

$$H^{(N)} = \mathcal{H}\left(\sum_{i=1}^N k_{2,i}\, \mathrm{e}^{p_i}, \sum_{i=1}^N k_{2,i}\, p_i\, \mathrm{e}^{p_i}, \sum_{i=1}^N \frac{k_{2,i}}{2}\, \mathrm{e}^{p_i}\left(p_i^2 + k_{3,i}\right), \sum_{i=1}^N \frac{k_{2,i}^c}{k_{1,i}}\, \mathrm{e}^{cp_i}, -\sum_{i=1}^N q_i\right). \tag{6.28}$$

Again, from the following nonlinear Casimirs we get the constants of the motion for this system:

$$C_1 = \frac{J_1^c}{J_4}, \qquad C_2 = J_1\, \mathrm{e}^{-\frac{J_2}{J_1}}, \qquad C_3 = \frac{2J_3}{J_1} - \frac{J_2^2}{J_1^2}, \tag{6.29}$$

$$C_1^{(m)} = \frac{\left(\sum_{i=1}^m k_{2,i}\, \mathrm{e}^{p_i}\right)^c}{\left(\sum_{i=1}^m \frac{k_{2,i}^c}{k_{1,i}}\, \mathrm{e}^{cp_i}\right)}, \qquad C_2^{(m)} = \left(\sum_{i=1}^m k_{2,i}\, \mathrm{e}^{p_i}\right) \mathrm{e}^{-\frac{(\sum_{i=1}^m k_{2,i}\, p_i\, \mathrm{e}^{p_i})}{(\sum_{i=1}^m k_{2,i}\, \mathrm{e}^{p_i})}}, \qquad m = 2, \ldots, N, \tag{6.30}$$

$$C_3^{(m)} = \frac{\left(\sum_{i=1}^m k_{2,i}\, \mathrm{e}^{p_i}\left(p_i^2 + k_{3,i}\right)\right)}{\left(\sum_{i=1}^m k_{2,i}\, \mathrm{e}^{p_i}\right)} - \frac{\left(\sum_{i=1}^m k_{2,i}\, p_i\, \mathrm{e}^{p_i}\right)^2}{\left(\sum_{i=1}^m k_{2,i}\, \mathrm{e}^{p_i}\right)^2}, \qquad m = 2, \ldots, N. \tag{6.31}$$

- $A_{5,12}$ *integrable systems.*

$$[J_1, J_5] = J_1, \quad [J_2, J_5] = J_1 + J_2, \quad [J_3, J_5] = J_2 + J_3, \quad [J_4, J_5] = J_3 + J_4, \tag{6.32}$$

$$\begin{aligned} H^{(N)} = \mathcal{H}\Bigg(&\sum_{i=1}^N k_{1,i}\, \mathrm{e}^{p_i}, \sum_{i=1}^N k_{1,i}\, p_i\, \mathrm{e}^{p_i}, \sum_{i=1}^N \frac{k_{1,i}}{2}\, \mathrm{e}^{p_i}\left(p_i^2 + k_{2,i}\right), \\ &\sum_{i=1}^N \frac{k_{1,i}\, \mathrm{e}^{p_i}}{6}\left(2k_{3,i} + 3k_{2,i}\, p_i + p_i^3\right), -\sum_{i=1}^N q_i\Bigg). \end{aligned} \tag{6.33}$$

Nonlinear Casimirs and their associated constants read

$$C_1 = J_1\, \mathrm{e}^{-\frac{J_2}{J_1}}, \qquad C_2 = \frac{2J_3}{J_1} - \frac{J_2^2}{J_1^2}, \qquad C_3 = \frac{3J_4}{J_1} - \frac{3J_2J_3}{J1^2} + \frac{J_2^3}{J_1^3}, \tag{6.34}$$

$$C_1^{(m)} = \left(\sum_{i=1}^m k_{1,i}\, \mathrm{e}^{p_i}\right) \mathrm{e}^{-\frac{(\sum_{i=1}^m k_{1,i}\, p_i\, \mathrm{e}^{p_i})}{(\sum_{i=1}^m k_{1,i}\, \mathrm{e}^{p_i})}}, \tag{6.35}$$

$$C_2^{(m)} = \frac{\left(\sum_{i=1}^m k_{1,i}\,\mathrm{e}^{p_i}\left(p_i^2 + k_{2,i}\right)\right)}{\left(\sum_{i=1}^m k_{1,i}\,\mathrm{e}^{p_i}\right)} - \frac{\left(\sum_{i=1}^m k_{1,i}\,p_i\,\mathrm{e}^{p_i}\right)^2}{\left(\sum_{i=1}^m k_{1,i}\,\mathrm{e}^{p_i}\right)^2}, \tag{6.36}$$

$$\begin{aligned} C_3^{(m)} = {} & \frac{1}{2}\frac{\left(\sum_{i=1}^m k_{1,i}\,\mathrm{e}^{p_i}\left(2k_{3,i} + 3k_{2,i}\,p_i + p_i^3\right)\right)}{\left(\sum_{i=1}^m k_{1,i}\,\mathrm{e}^{p_i}\right)} \\ & - \frac{3}{2}\frac{\left(\sum_{i=1}^m k_{1,i}\,p_i\,\mathrm{e}^{p_i}\right)\left(\sum_{i=1}^m k_{1,i}\,\mathrm{e}^{p_i}\left(p_i^2 + k_{2,i}\right)\right)}{\left(\sum_{i=1}^m k_{1,i}\,\mathrm{e}^{p_i}\right)^2} + \frac{\left(\sum_{i=1}^m k_{2,i}\,p_i\,\mathrm{e}^{p_i}\right)^3}{\left(\sum_{i=1}^m k_{1,i}\,\mathrm{e}^{p_i}\right)^3}. \end{aligned} \tag{6.37}$$

- $A_{5,15}^{\alpha}$ *integrable systems.*

$$[J_1, J_5] = J_1, \quad [J_2, J_5] = J_1 + J_2, \quad [J_3, J_5] = \alpha J_3, \quad [J_4, J_5] = J_3 + \alpha J_4 \quad (|\alpha| \leqslant 1), \tag{6.38}$$

$$\begin{aligned} H^{(N)} = \mathcal{H}\Bigg(& \sum_{i=1}^N \mathrm{e}^{p_i}, \sum_{i=1}^N (p_i - \log k_{2,i})\,\mathrm{e}^{p_i}, \\ & \sum_{i=1}^N \frac{\mathrm{e}^{\alpha p_i}}{k_{1,i}}, \sum_{i=1}^N \frac{\mathrm{e}^{\alpha p_i}}{\alpha k_{1,i}}\left(\alpha p_i - \log k_{1,i}k_{3,i}\right), -\sum_{i=1}^N q_i \Bigg). \end{aligned} \tag{6.39}$$

Finally, the complete integrability is given by the following nonlinear Casimirs:

$$C_1 = \frac{J_1^{\alpha}}{J_3}, \qquad C_2 = J_1\,\mathrm{e}^{-\frac{J_2}{J_1}}, \qquad C_3 = J_3\,\mathrm{e}^{-\alpha\frac{J_4}{J_3}}, \tag{6.40}$$

$$C_1^{(m)} = \frac{\left(\sum_{i=1}^m \mathrm{e}^{p_i}\right)^{\alpha}}{\left(\sum_{i=1}^m \frac{\mathrm{e}^{\alpha p_i}}{k_{1,i}}\right)}, \qquad C_2^{(m)} = \left(\sum_{i=1}^m \mathrm{e}^{p_i}\right)\mathrm{e}^{-\frac{(\sum_{i=1}^m (p_i - \log k_{2,i})\,\mathrm{e}^{p_i})}{(\sum_{i=1}^m \mathrm{e}^{p_i})}}, \qquad m = 2, \ldots, N, \tag{6.41}$$

$$C_3^{(m)} = \left(\sum_{i=1}^m \frac{\mathrm{e}^{\alpha p_i}}{k_{1,i}}\right)\cdot\mathrm{e}^{-\frac{(\sum_{i=1}^m \frac{\mathrm{e}^{\alpha p_i}}{k_{1,i}}(\alpha p_i - \log k_{1,i}k_{3,i}))}{(\sum_{i=1}^m \frac{\mathrm{e}^{\alpha p_i}}{k_{1,i}})}}, \qquad m = 2, \ldots, N. \tag{6.42}$$

7. Concluding remarks

In higher dimensions, classifications of real Lie algebras and their Casimir invariants are partial and restricted to certain simple, solvable or nilpotent subclasses, but a significant number of the latter are known (see, for instance, [22–28]). Thus, the method presented here can be used to generate many new families of integrable systems provided that, for a given Lie coalgebra with known Casimir invariants, the integrability criterion is checked in order to determine *a priori* which are the symplectic realizations that can lead to integrable systems.

It is also interesting to stress that symplectic realizations that do not fulfil the integrability condition can also lead to interesting (but partially integrable) Hamiltonian models. This is the case of the 'two-photon' algebra h_6, a 6D Lie algebra with $r = 2$ (therefore $s = 1, 2$) that admits an $s = 1$ symplectic realization [4] for which, among the $2N$ integrals provided by the coalgebra, only $(2N - 5)$ of them are functionally independent and $(N - 2)$ are in involution. Hence, any Hamiltonian $\mathcal{H}$ with h_6-coalgebra symmetry is 'almost' integrable (only one constant is left), and such a remaining integral does exist for some special choices of $\mathcal{H}$ which can be in some cases connected with the subalgebras of h_6 [4].

Finally, we would like to mention that the integrability conditions presented here can be generalized to the case in which symplectic realizations with different dimension *s* are used on each copy of the algebra. It would be also interesting to perform a detailed analysis of some of the new Hamiltonians presented here, working out explicit the solutions for the equations of motion. We recall that the latter problem can always be faced through the cluster variables technique [5] that makes use of the coalgebra symmetry in order to define the appropriate collective dynamical variables. Work on all these lines is in progress.

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